

Necessary and sufficient Tauberian conditions for the logarithmic summability of functions and sequences

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Abstract

Let $s : [1, \infty) \rightarrow \mathbb{C}$ be a locally integrable function in Lebesgue's sense on the infinite interval $[1, \infty)$. We say that s is summable $(L, 1)$ if there exists some $A \in \mathbb{C}$ such that

$$(*) \quad \lim_{t \rightarrow \infty} \tau(t) = A, \quad \text{where} \quad \tau(t) := \frac{1}{\log t} \int_1^t \frac{s(u)}{u} du.$$

It is clear that if the ordinary limit $s(t) \rightarrow A$ exists, then the limit $\tau(t) \rightarrow A$ also exists as $t \rightarrow \infty$. We present sufficient conditions, which are also necessary in order that the converse implication hold true. As corollaries, we obtain so-called Tauberian theorems which are analogous to those known in the case of summability $(C, 1)$. For example, if the function s is slowly oscillating, by which we mean that for every $\varepsilon > 0$ there exist $t_0 = t_0(\varepsilon) > 1$ and $\lambda = \lambda(\varepsilon) > 1$ such that

$$|s(u) - s(t)| \leq \varepsilon \quad \text{whenever} \quad t_0 \leq t < u \leq t^\lambda,$$

then the converse implication holds true: the ordinary convergence $\lim_{t \rightarrow \infty} s(t) = A$ follows from $(*)$.

We also present necessary and sufficient Tauberian conditions under which the ordinary convergence of a numerical sequence (s_k) follows from its logarithmic summability. Among others, we give a more transparent proof of an earlier Tauberian theorem due to Kwee [3].

1. *Introduction: Summability $(C, 1)$ and $(L, 1)$ of functions*

Let $s : [0, \infty) \rightarrow \mathbb{C}$ be an integrable function in Lebesgue's sense on every bounded interval $[0, t]$, $t > 0$, in symbols: $s \in L_{\text{loc}}[0, \infty)$. We recall (see, e.g., [2, p. 11]) that the function s is said to be Cesàro summable of first order, briefly: summable $(C, 1)$, if there exists some $A \in \mathbb{C}$ such that

$$(1.1) \quad \lim_{t \rightarrow \infty} \sigma(t) = A, \quad \text{where} \quad \sigma(t) := \frac{1}{t} \int_0^t s(u) du, \quad t > 0.$$

Clearly, if the ordinary limit

$$(1.2) \quad \lim_{t \rightarrow \infty} s(t) = A$$

exists, then the limit in (1.1) also exists. The converse implication holds true only under some supplementary, so-called Tauberian condition(s).

We note that the left endpoint of the definition domain of the function s is indifferent in (1.1). That is, given any $a > 0$, the existence of the limit in (1.1) is equivalent with the existence of the following one:

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_a^t s(u) du = A.$$

As to the Cesàro summability of order α , where $\alpha \geq 0$ is a real number, briefly: summability (C, α) , we refer to [7, p. 26]. The case $\alpha = 0$ is ordinary convergence.

Next, let $s : [1, \infty) \rightarrow \mathbb{C}$ be such that $s \in L_{\text{loc}}[1, \infty)$. Motivated by the concept of logarithmic (sometimes also called harmonic) summability of a numerical sequence (see, e.g., in [5]), the function s is said to be logarithmic summable of first order, briefly: summable $(L, 1)$, if there exists some $A \in \mathbb{C}$ such that

$$(1.3) \quad \lim_{t \rightarrow \infty} \tau(t) = A, \quad \text{where} \quad \tau(t) := \frac{1}{\log t} \int_1^t \frac{s(u)}{u} du, \quad t > 1,$$

where the logarithm is to the natural base e .

In Section 4, we will prove that summability $(C, 1)$ of a function implies its summability $(L, 1)$ to the same limit, but the converse implication is not true in general.

We note that a complex-valued function $s \in L_{\text{loc}}[e, \infty)$ is said to be logarithmic summable of order 2, briefly: summable $(L, 2)$, if there exists some $A \in \mathbb{C}$ such that

$$(1.4) \quad \lim_{t \rightarrow \infty} \tau_2(t) = A \quad \text{where} \quad \tau_2(t) := \frac{1}{\log \log t} \int_e^t \frac{s(u)}{u \log u} du, \quad t > e.$$

We also note that in the particular cases when

$$(1.5) \quad s(u) := \int_0^u f(x)dx, \quad u > 0; \quad \text{or} \quad s(u) := \int_1^u f(x)dx, \quad u > 1;$$

where f is a locally integrable function on $[0, \infty)$ or $[1, \infty)$, respectively, the above summability methods may be applied to assign value to the following integrals, respectively:

$$(1.6) \quad \int_0^\infty f(x)dx \quad \text{or} \quad \int_1^\infty f(x)dx.$$

If the finite limit in (1.2) exists, then the improper integrals $\int_0^{\rightarrow\infty} f(x)dx$ and $\int_1^{\rightarrow\infty} f(x)dx$ exist, respectively. In the case when the finite limit in (1.1) exists, then the integral in (1.6) (i) is said to be summable $(C, 1)$; while in the case when only the finite limit in (1.3) exists, then the integral in (1.6) (ii) is said to be summable $(L, 1)$.

2. Main results

In our first new result we characterize the converse implication when the ordinary limit of a real-valued function at ∞ follows from its summability $(L, 1)$.

THEOREM 1. *If a real-valued function $s \in L_{\text{loc}}[1, \infty)$ is summable $(L, 1)$ to some $A \in \mathbb{R}$, then the ordinary limit (1.2) exists if and only if*

$$(2.1) \quad \limsup_{\lambda \rightarrow 1+} \liminf_{t \rightarrow \infty} \frac{1}{(\lambda - 1) \log t} \int_t^{t^\lambda} \frac{s(u) - s(t)}{u} du \geq 0$$

and

$$(2.2) \quad \limsup_{\lambda \rightarrow 1-} \liminf_{t \rightarrow \infty} \frac{1}{(1 - \lambda) \log t} \int_{t^\lambda}^t \frac{s(t) - s(u)}{u} du \geq 0.$$

Motivated by the definition of the ‘slow decrease’ with respect to summability $(C, 1)$ (see, e.g., [2, pp. 124-125; and cf. our Remark 1 below]), we say that a function $s : [1, \infty) \rightarrow \mathbb{R}$ is slowly decreasing with respect to summability $(L, 1)$ if for every $\varepsilon > 0$ there exist $t_0 = t_0(\varepsilon) > 1$ and $\lambda = \lambda(\varepsilon) > 1$ such that

$$(2.3) \quad s(u) - s(t) \geq -\varepsilon \quad \text{whenever} \quad t_0 \leq t < u \leq t^\lambda.$$

It is easy to check that a function s is slowly decreasing with respect to summability $(L, 1)$ if and only if

$$(2.4) \quad \lim_{\lambda \rightarrow 1+} \liminf_{t \rightarrow \infty} \inf_{t < u \leq t^\lambda} (s(u) - s(t)) \geq 0.$$

Since the auxiliary function

$$a(\lambda) := \liminf_{t \rightarrow \infty} \inf_{t < u \leq t^\lambda} (s(u) - s(t))$$

is evidently decreasing in λ on the infinite interval $(1, \infty)$, the right limit in (2.4) exists and $\lim_{\lambda \rightarrow 1+}$ can be replaced by $\sup_{\lambda > 1}$.

It is clear that if a function $s \in L_{\text{loc}}[1, \infty)$ is slowly decreasing with respect to summability $(L, 1)$, then conditions (2.1) and (2.2) are trivially satisfied. Thus, the next corollary is an immediate consequence of Theorem 1.

COROLLARY 1. *Suppose a real-valued function $s \in L_{\text{loc}}[1, \infty)$ is slowly decreasing with respect to summability $(L, 1)$. If s is summable $(L, 1)$ to some $A \subset \mathbb{R}$, then the ordinary limit (1.2) also exists.*

Historically, the term ‘slow decrease’ was introduced by Schmidt [6] in the case of the summability $(C, 1)$ of sequences of real numbers.

In our second new result we characterize the converse implication when the ordinary convergence of a complex-valued function follows from its summability $(L, 1)$.

THEOREM 2. *If a complex-valued function $s \in L_{\text{loc}}[1, \infty)$ is summable $(L, 1)$ to some $A \in \mathbb{C}$, then the ordinary limit (1.2) exists if and only if*

$$(2.5) \quad \liminf_{\lambda \rightarrow 1+} \limsup_{t \rightarrow \infty} \left| \frac{1}{(\lambda - 1) \log t} \int_t^{t^\lambda} \frac{s(u) - s(t)}{u} du \right| = 0.$$

Motivated by the definition of the ‘slow oscillation’ with respect to summability $(C, 1)$ of numerical sequences introduced by Hardy [1] (see also in [2, pp. 124-125]), we say that a function $s : [1, \infty) \rightarrow \mathbb{C}$ is slowly oscillating with respect to summability $(L, 1)$ if for every $\varepsilon > 0$ there exist $t_0 = t_0(\varepsilon) > 1$ and $\lambda = \lambda(\varepsilon) > 1$ such that

$$(2.6) \quad |s(u) - s(t)| \leq \varepsilon \quad \text{whenever} \quad t_0 \leq t < u \leq t^\lambda.$$

It is easy to check that a function s is slowly oscillating with respect to summability $(L, 1)$ if and only if

$$(2.7) \quad \lim_{\lambda \rightarrow 1+} \limsup_{t \rightarrow \infty} \sup_{t < u \leq t\lambda} |s(u) - s(t)| = 0.$$

It is clear that if a function $s \in L_{\text{loc}}[1, \infty)$ is slowly oscillating with respect to summability $(L, 1)$, then condition (2.5) is trivially satisfied. Thus, the next corollary is an immediate consequence of Theorem 2.

COROLLARY 2. *Suppose a complex-valued function $s \in L_{\text{loc}}[1, \infty)$ is slowly oscillating with respect to summability $(L, 1)$. If s is summable $(L, 1)$ to some $A \in \mathbb{C}$. then the ordinary limit (1.2) also exists.*

REMARK 1. According to Hardy's definition (see [2, pp. 124-125]), a function $s : (0, \infty) \rightarrow \mathbb{C}$ is said to be slowly oscillating if

$$(2.8) \quad \lim(s(u) - s(t)) = 0 \quad \text{whenever} \quad u > t \rightarrow \infty \quad \text{and} \quad u/t \rightarrow 1;$$

and a function $s : (0, \infty) \rightarrow \mathbb{R}$ is said to be slowly decreasing if

$$(2.9) \quad \liminf(s(u) - s(t)) \geq 0 \quad \text{under the same circumstances.}$$

We claim that definition (2.8) is equivalent to the following one: for every $\varepsilon > 0$ there exist $t_0 = t_0(\varepsilon) > 0$ and $\lambda = \lambda(\varepsilon) > 1$ such that

$$(2.10) \quad |s(u) - s(t)| \leq \varepsilon \quad \text{whenever} \quad t_0 \leq t < u \leq \lambda t.$$

The implication (2.8) \Rightarrow (2.10) is trivial. To justify the converse implication (2.10) \Rightarrow (2.8), let $\lambda > 1$ be arbitrarily close to 1 and set $\varepsilon := \log \lambda$. Then by (2.10), we have

$$|s(u) - s(t)| \leq \varepsilon \quad \text{whenever} \quad u > t \geq t_0 \quad \text{and} \quad 0 < \log \frac{u}{t} \leq \log \lambda = \varepsilon.$$

Now, the equivalence of the two definitions claimed above is obvious.

It is worth to consider the special case (1.6) (ii), where $f \in L_{\text{loc}}[1, \infty)$. If f is a real-valued function and

$$(2.11) \quad x(\log x)f(x) \geq -C \quad \text{at almost every} \quad x > x_0,$$

where $C > 0$ and $x_0 \geq 1$ are constants, then s defined in (1.5) (ii) is slowly decreasing with respect to summability $(L, 1)$, and Corollary 1 applies. Likewise, if f is a complex-valued function and

$$(2.12) \quad x(\log x)|f(x)| \leq C \quad \text{at almost every } x > x_0,$$

where $C > 0$ and $x_0 \geq 1$ are constants, then s is slowly oscillating with respect to summability $(L, 1)$, and Corollary 2 applies.

Condition (2.11) is called a one-sided Tauberian condition, while (2.12) is called a two-sided Tauberian condition with respect to summability $(L, 1)$. These terms go back to Landau [4] with respect to summability $(C, 1)$ of sequences of real numbers, as well as to Hardy [1] (see also [2, p. 149]) with respect to summability $(C, 1)$ of sequences of complex numbers.

We note that such theorems containing appropriate additional conditions such as (2.11), (2.12), etc. are called ‘Tauberian’, after A. Tauber, who first proved one of the simplest of this kind; and these supplementary conditions are called ‘Tauberian conditions’.

3. Proofs of Theorems 1 and 2

The following two representations of the difference $s(t) - \tau(t)$ will be of vital importance in our proofs below.

LEMMA 1. (i) If $\lambda > 1$ and $t > 1$, then

$$(3.1) \quad s(t) - \tau(t) = \frac{\lambda}{\lambda - 1}(\tau(t^\lambda) - \tau(t)) - \frac{1}{(\lambda - 1)\log t} \int_t^{t^\lambda} \frac{s(u) - s(t)}{u} du.$$

(ii) If $0 < \lambda < 1$ and $t > 1$, then

$$(3.2) \quad s(t) - \tau(t) = \frac{\lambda}{1 - \lambda}(\tau(t) - \tau(t^\lambda)) + \frac{1}{(1 - \lambda)\log t} \int_{t^\lambda}^t \frac{s(t) - s(u)}{u} du.$$

Proof. Part (i). By definition in (1.3), we have

$$\begin{aligned} \tau(t^\lambda) - \tau(t) &= \frac{1}{\lambda \log t} \int_1^{t^\lambda} \frac{s(u)}{u} du - \frac{1}{\log t} \int_1^t \frac{s(u)}{u} du \\ &= \frac{1 - \lambda}{\lambda \log t} \int_1^t \frac{s(u)}{u} du + \frac{1}{\lambda \log t} \int_t^{t^\lambda} \frac{s(u)}{u} du \end{aligned}$$

$$= -\frac{\lambda-1}{\lambda}\tau(t) + \frac{1}{\lambda \log t} \int_t^{t^\lambda} \frac{s(u)}{u} du.$$

Multiplying both sides by $\lambda/(\lambda-1)$ gives

$$\begin{aligned} \frac{\lambda}{\lambda-1}(\tau(t^\lambda) - \tau(t)) &= -\tau(t) + \frac{1}{(\lambda-1) \log t} \int_t^{t^\lambda} \frac{s(u)}{u} du \\ &= s(t) - \tau(t) + \frac{1}{(\lambda-1) \log t} \int_t^{t^\lambda} \frac{s(u) - s(t)}{u} du, \end{aligned}$$

whence (3.1) follows.

Part (ii). The proof of (3.2) is analogous to that of (3.1).

Proof of Theorem 1. Necessity. Suppose that (1.2) is satisfied. By (1.2) and (1.3), we have

$$(3.3) \quad \lim_{t \rightarrow \infty} (s(t) - \tau(t)) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} (\tau(t^\lambda) - \tau(t)) = 0$$

for each fixed $\lambda > 1$. By (3.1) and (3.3), we conclude that

$$(3.4) \quad \lim_{t \rightarrow \infty} \frac{1}{\log t} \int_t^{t^\lambda} \frac{s(u) - s(t)}{u} du = 0$$

for every $\lambda > 1$. This proves (2.1) even in a stronger form.

An analogous argument yields (2.2) for every $0 < \lambda < 1$ also in a stronger form.

Sufficiency. Suppose that (2.1) and (2.2) are satisfied. By (2.1), there exists a sequence $\lambda_j \downarrow 1$ such that

$$(3.5) \quad \lim_{j \rightarrow \infty} \liminf_{t \rightarrow \infty} \frac{1}{(\lambda_j - 1) \log t} \int_t^{t^{\lambda_j}} \frac{s(u) - s(t)}{u} du \geq 0.$$

By (1.3), (3.1) and (3.5), we conclude that

$$\begin{aligned} (3.6) \quad \limsup_{t \rightarrow \infty} (s(t) - \tau(t)) &\leq \lim_{j \rightarrow \infty} \limsup_{t \rightarrow \infty} \frac{\lambda_j}{\lambda_j - 1} (\tau(t^{\lambda_j}) - \tau(t)) \\ &\quad + \lim_{j \rightarrow \infty} \limsup_{t \rightarrow \infty} \left(-\frac{1}{(\lambda_j - 1) \log t} \int_t^{t^{\lambda_j}} \frac{s(u) - s(t)}{u} du \right) \\ &= -\lim_{j \rightarrow \infty} \liminf_{t \rightarrow \infty} \frac{1}{(\lambda_j - 1) \log t} \int_t^{t^{\lambda_j}} \frac{s(u) - s(t)}{u} du \leq 0. \end{aligned}$$

By (2.2), there exists a sequence $0 < \lambda_k \uparrow 1$ such that

$$(3.7) \quad \lim_{k \rightarrow \infty} \liminf_{t \rightarrow \infty} \frac{1}{(1 - \lambda_k) \log t} \int_{t^{\lambda_k}}^t \frac{s(t) - s(u)}{u} du \geq 0.$$

By (1.3), (3.2) and (3.7), we conclude that

$$(3.8) \quad \begin{aligned} \liminf_{t \rightarrow \infty} (s(t) - \tau(t)) &\geq \lim_{k \rightarrow \infty} \liminf_{t \rightarrow \infty} \frac{\lambda_k}{1 - \lambda_k} (\tau(t) - (\tau(t^{\lambda_k}))) \\ &\quad + \lim_{k \rightarrow \infty} \liminf_{t \rightarrow \infty} \frac{1}{(1 - \lambda_k) \log t} \int_{t^{\lambda_k}}^t \frac{s(t) - s(u)}{u} du \\ &= \lim_{k \rightarrow \infty} \liminf_{t \rightarrow \infty} \frac{1}{(1 - \lambda_k) \log t} \int_{t^{\lambda_k}}^t \frac{s(t) - s(u)}{u} du \geq 0. \end{aligned}$$

Combining (3.6) and (3.8) yields (3.3) (i), and a fortiori, we get (1.2) to be proved, due to summability $(L, 1)$ of the function s .

The proof of Theorem 1 is complete.

Proof of Theorem 2. It also hinges on Lemma 1 and runs along similar lines to the proof of Theorem 1. The details are left to the reader.

4. Inclusions

We will prove that summability $(L, 1)$ is more effective than summability $(C, 1)$.

THEOREM 3. *If a complex-valued function $s \in L_{\text{loc}}[1, \infty)$ is summable $(C, 1)$ to some $A \in \mathbb{C}$, then s is also summable $(L, 1)$ to the same A . The converse implication is not true in general.*

Proof. (i) First, let $t := m$, where $m = 2, 3, \dots$. By definition in (1.4) and applying the Second Mean-Value Theorem, we get

$$(4.1) \quad \begin{aligned} \tau(m) \log m &= \sum_{k=1}^{m-1} \int_k^{k+1} \frac{s(u)}{u} du \\ &= \sum_{k=1}^{m-1} \left(\frac{1}{k} \int_k^{\xi_k} s(u) du + \frac{1}{k+1} \int_{\xi_k}^{k+1} s(u) du \right) \end{aligned}$$

$$= \int_1^{\xi_1} s(u)du + \sum_{k=1}^{m-2} \frac{1}{k+1} \int_{\xi_k}^{\xi_{k+1}} s(u)du + \frac{1}{m} \int_{\xi_{m-1}}^m s(u)du,$$

where

$$(4.2) \quad k < \xi_k < k+1 \quad \text{for} \quad k = 1, 2, \dots, m-1.$$

By definition in (1.1), we may write that

$$(4.3) \quad \int_{\xi}^{\eta} s(u)du = \eta\sigma(\eta) - \xi\sigma(\xi), \quad 0 < \xi < \eta.$$

Making use of this equality, from (4.1) it follows that

$$\begin{aligned} \tau(m) \log m &= -(\xi_1\sigma(\xi_1) - \sigma(1)) \\ &+ \sum_{k=1}^{m-2} \frac{1}{k+1} (\xi_{k+1}\sigma(\xi_{k+1}) - \xi_k\sigma(\xi_k)) + \frac{1}{m} (m\sigma(m) - \xi_{m-1}\sigma(\xi_{m-1})) \\ &= -\sigma(1) + \sum_{k=1}^{m-1} \frac{1}{k(k+1)} \xi_k\sigma(\xi_k) + \sigma(m) - \sigma(1), \end{aligned}$$

whence we get

$$(4.4) \quad \tau(m) = \frac{1}{\log m} \sum_{k=1}^{m-1} \frac{\xi_k}{k(k+1)} \sigma(\xi_k) + \frac{1}{\log m} (\sigma(m) - \sigma(1)).$$

We will apply Toeplitz' theorem on the summability of numerical sequences (see, e.g., [8, p. 74]) in the case of (4.4) with the infinite triangular matrix

$$\left(a_{m,k} := \frac{1}{\log m} \frac{\xi_k}{k(k+1)}, \quad k = 1, 2, \dots, m-1; m = 2, 3, \dots \right).$$

By (4.2), we have

$$\frac{1}{\log m} \sum_{k=1}^{m-1} \frac{1}{k+1} < \sum_{k=1}^{m-1} a_{m,k} < \frac{1}{\log m} \sum_{k=1}^{m-1} \frac{1}{k}, \quad m = 2, 3, \dots;$$

whence it follows that

$$\lim_{m \rightarrow \infty} \sum_{k=1}^{m-1} a_{m,k} = 1.$$

It is also clear that

$$0 < a_{m,k} < \frac{1}{k \log m} \rightarrow 0 \quad \text{as } m \rightarrow \infty \quad \text{for } k = 1, 2, \dots$$

Thus, the sufficient conditions are satisfied in Toeplitz' theorem, and we conclude that the limit in (1.3) holds in the particular choice when $t = m \in \mathbb{N}$.

(ii) Second, given any real number $t > 3$, let $m := [t]$, the integer part of t . We use (4.3) and the Second Mean-Value Theorem again to obtain

$$\begin{aligned} (4.5) \quad \tau(t) \log t - \tau(m) \log m &= \int_m^t \frac{s(u)}{u} du \\ &= \frac{1}{m} \int_m^\xi s(u) du - \frac{1}{t} \int_\xi^t s(u) du \\ &= \left(\frac{1}{m} - \frac{1}{t} \right) \xi \sigma(\xi) - \sigma(m) + \sigma(t), \quad m < \xi < t. \end{aligned}$$

By (1.1) and (4.5), we get

$$\begin{aligned} |\tau(t) \log t - \tau(m) \log m| &= \frac{t-m}{mt} \xi |\sigma(\xi)| + |\sigma(t) - \sigma(m)| \\ &\leq \frac{1}{m} |\sigma(\xi)| + |\sigma(t) - \sigma(m)| \rightarrow 0 \quad \text{as } m \rightarrow \infty. \end{aligned}$$

Hence we conclude that

$$\lim_{t \rightarrow \infty} \tau(t) = \lim_{t \rightarrow \infty} \tau(m) \frac{\log m}{\log t} = A, \quad \text{where } m := [t].$$

(iii) Third, to see that the converse implication is not true in general, we consider the function s defined by

$$s(t) := \begin{cases} me^{2^m} & \text{if } t \in [e^{2^m}, e^{2^m} + 1], m = 1, 2, \dots; \\ 0 & \text{otherwise on } [1, \infty). \end{cases}$$

We claim that this function s cannot be summable $(C, 1)$ to any finite number A . To this effect, we recall that if we had (1.1), then for any number $a > 0$ we would have

$$\frac{1}{t} \int_t^{t+a} s(u) du = \frac{t+a}{t} \sigma(t+a) - \sigma(t) \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

But for $t := e^{2^m}$ and $a := 1$, we have

$$\frac{1}{t} \int_t^{t+1} s(u) du = e^{-2^m} \int_t^{t+1} me^{2^m} du = m \not\rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Consequently, for this function s the limit (1.1) cannot exist with any finite number A .

On the other hand, if t is such that

$$e^{2^{m-1}} \leq t < e^{2^m}, \quad m = 1, 2, \dots;$$

then we estimate as follows:

$$\begin{aligned} 0 \leq \tau(t) &\leq \frac{1}{2^{m-1}} \sum_{k=1}^{m-1} \int_{e^{2^{k-1}}}^{e^{2^k}} \frac{s(u)}{u} du \\ &\leq \frac{1}{2^{m-1}} \sum_{k=1}^{m-1} k \rightarrow 0 \quad \text{as } t \rightarrow \infty. \end{aligned}$$

This proves that the limit in (1.3) exists with $A = 0$.

The proof of Theorem 3 is complete.

We note that summability $(L, 2)$ is more effective than summability $(L, 1)$. This can be proved in an analogous way as Theorem 3 was proved above. We refer to [5, on p. 382], where an analogous result is proved for the logarithmic mean $\tau_2(n)$ of second order of a numerical sequence (s_k) (see also (5.2) below).

5. Summability $(L, 1)$ of numerical sequences

The above methods of summability are the nondiscrete ones of the methods of logarithmic summability of numerical sequences $(s_k) = (s_k : k = 1, 2, \dots)$ of complex numbers. We recall that a sequence (s_k) is said to be logarithmic summable of order 1 (see in [5], where the term ‘harmonic summable of order 1’ was used), briefly: summable $(L, 1)$, if there exists some $A \in \mathbb{C}$ such that

$$(5.1) \quad \lim_{n \rightarrow \infty} \frac{1}{\ell_n} \sum_{k=1}^n \frac{s_k}{k} = A, \quad \text{where } \ell_n := \sum_{k=1}^n \frac{1}{k} \sim \log n,$$

where for two sequences (a_n) and (b_n) of positive numbers we write $a_n \sim b_n$ if

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 1.$$

We note that the sequence (s_k) is said to be logarithmic summable of order 2 (see also in [5]), briefly: summable $(L, 2)$, if there exists some $A \in \mathbb{C}$ such that

$$(5.2) \quad \lim_{n \rightarrow \infty} \tau_2(n) := \frac{1}{\ell_n(2)} \sum_{k=1}^n \frac{s_k}{k \ell_k}, \quad \text{where } \ell_n(2) := \sum_{k=1}^n \frac{1}{k \ell_k} \sim \log \log n.$$

It is clear that if the ordinary limit

$$(5.3) \quad \lim_{n \rightarrow \infty} s_n = A$$

exists, then the limit in (5.1) also exists with the same A . Even more is true (see, e.g., in [5, on p. 376]): If a sequence (s_k) is such that the finite limit

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n s_k = A$$

exists, then the limit in (5.1) also exists with the same A . The converse implication is not true in general.

We note that if the finite limit in (5.1) exists, then the limit in (5.2) also exists with the same A (see also in [5, on p. 382]. Again, the converse implication is not true in general.

Now, the discrete analogue of Theorem 1 reads as follows.

THEOREM 4. *If a sequence (s_k) of real numbers is summable $(L, 1)$ to some $A \in \mathbb{R}$, then the ordinary limit (5.3) exists if and only if*

$$(5.4) \quad \limsup_{\lambda \rightarrow 1+} \liminf_{n \rightarrow \infty} \frac{1}{([n^\lambda] - n)\ell_n} \sum_{k=n+1}^{[n^\lambda]} \frac{s_k - s_n}{k} \geq 0$$

and

$$(5.5) \quad \limsup_{\lambda \rightarrow 1-} \liminf_{n \rightarrow \infty} \frac{1}{(n - [n^\lambda])\ell_n} \sum_{k=[n^\lambda]+1}^n \frac{s_n - s_k}{k} \geq 0,$$

where by $[\cdot]$ we denote the integer part of a real number, and ℓ_n is defined in (5.1).

Analogously to (2.3), we say that a sequence (s_k) of real numbers is slowly decreasing with respect to summability $(L, 1)$ if for every $\varepsilon > 0$ there exist a natural number $n_0 = n_0(\varepsilon)$ and a real number $\lambda = \lambda(\varepsilon) > 1$ such that

$$(5.6) \quad s_k - s_n \geq -\varepsilon \quad \text{whenever} \quad n_0 \leq n < k \leq n^\lambda.$$

It is easy to check (cf. (2.4)) that a sequence (s_k) is slowly decreasing with respect to summability $(L, 1)$ if and only if

$$(5.7) \quad \lim_{\lambda \rightarrow 1+} \liminf_{n \rightarrow \infty} \min_{n < k \leq n^\lambda} (s_k - s_n) \geq 0.$$

Clearly, if a sequence (s_k) is slowly decreasing with respect to summability $(L, 1)$, then both conditions (5.4) and (5.5) are satisfied. Thus, the next corollary is an immediate consequence of Theorem 4.

COROLLARY 3. *Suppose a sequence (s_k) of real numbers is slowly decreasing with respect to summability $(L, 1)$. If (s_k) is summable $(L, 1)$ to some $A \in \mathbb{R}$, then the ordinary limit (5.3) also exists.*

This corollary was earlier proved by Kwee [3, Lemma 3] in a different way. We note that the definition of slow decrease of a sequence (s_k) is formally different in [3] from the definitions given in (5.6) and (5.7) above.

REMARK 2. According to Kwee's definition in [3, see it as a condition in both Theorem A and Lemma 3], a sequence of real numbers (s_k) is said to be slowly decreasing if

$$(5.8) \quad \liminf(s_k - s_n) \geq 0 \quad \text{whenever} \quad k > n \rightarrow \infty \quad \text{and} \quad \frac{\log k}{\log n} \rightarrow 1.$$

We claim that definition (5.8) is equivalent to the one in (5.6) (as well as to the one in (5.7)). The implication $(5.8) \Rightarrow (5.6)$ is trivial. To justify the converse implication $(5.8) \Rightarrow (5.6)$, let $\lambda > 1$ be arbitrarily close to 1 and set $\varepsilon := \log \lambda$. By (5.8), there exists $n_0 = n_0(\lambda) > 1$ such that

$$s_k - s_n \geq -\varepsilon \quad \text{whenever} \quad k > n \geq n_0 \quad \text{and} \quad 0 < \log \frac{\log k}{\log n} \leq \log \lambda = \varepsilon.$$

Now, the equivalence of the two definitions claimed above is obvious.

Next, the discrete analogue of Theorem 2 reads as follows.

THEOREM 5. *If a sequence (s_n) of complex numbers is summable $(L, 1)$ to some $A \in \mathbb{C}$, then the ordinary limit (5.3) exists if and only if*

$$(5.9) \quad \lim_{\lambda \rightarrow 1+} \limsup_{t \rightarrow \infty} \left| \frac{1}{([n^\lambda] - n)\ell_n} \sum_{k=n+1}^{[n^\lambda]} \frac{s_k - s_n}{k} \right| = 0.$$

Analogously to (2.6), we say that a sequence (s_k) of complex numbers is slowly oscillating with respect to summability $(L, 1)$ if for every $\varepsilon > 0$ there exist $n_0 = n_0(\varepsilon) > 1$ and $\lambda = \lambda(\varepsilon) > 1$ such that

$$(5.10) \quad |s_k - s_n| \leq \varepsilon \quad \text{whenever} \quad n_0 \leq n < k \leq n^\lambda.$$

It is easy to check that a sequence (s_k) is slowly oscillating with respect to summability $(L, 1)$ if and only if

$$(5.11) \quad \lim_{\lambda \rightarrow 1+} \limsup_{n \rightarrow \infty} \max_{n < k \leq n^\lambda} |s_k - s_n| = 0.$$

REMARK 3. The concept of slow oscillation with respect to summability $(L, 1)$ is not defined in [3]. However, analogously to (5.8), a sequence (s_k) of complex numbers may be called to be slowly oscillating if

$$(5.12) \quad \lim(s_k - s_n) = 0 \quad \text{whenever} \quad k > n \rightarrow \infty \quad \text{and} \quad \frac{\log k}{\log n} \rightarrow 1.$$

A reasoning similar to the one in Remark 2 gives that the definitions (5.12) and (5.10) (as well as (5.11)) are equivalent.

It is clear that if a sequence (s_k) is slowly oscillating with respect to $(L, 1)$, then condition (5.11) is satisfied. Thus, the next corollary is an immediate consequence of Theorem 5.

COROLLARY 4. *Suppose a sequence (s_k) of complex numbers is slowly oscillating with respect to summability $(L, 1)$. If (s_k) is summable $(L, 1)$ to some $A \in \mathbb{C}$, then the ordinary limit (5.3) also exists.*

The proofs of Theorems 4 and 5 run along similar lines to those of Theorems 1 and 2, respectively; while the key ingredient is provided by the following

LEMMA 2. (i) *For all $\lambda > 1$ and large enough n , that is when $[n^\lambda] > n$, we have the representation*

$$s_n - \tau_n = \frac{\ell_{[n^\lambda]} - \ell_n}{\ell_{[n^\lambda]} - \ell_n} (\tau_{[n^\lambda]} - \tau_n) - \frac{1}{\ell_{[n^\lambda]} - \ell_n} \sum_{k=n+1}^{[n^\lambda]} \frac{s_k - s_n}{k}.$$

(ii) *For all $0 < \lambda < 1$ and large enough n , that is when $n > [n^\lambda]$, we have*

$$s_n - \tau_n = \frac{\ell_{[n^\lambda]} - \ell_n}{\ell_n - \ell_{[n^\lambda]}} (\tau_n - \tau_{[n^\lambda]}) + \frac{1}{\ell_n - \ell_{[n^\lambda]}} \sum_{k=[n^\lambda]+1}^n \frac{s_n - s_k}{k}.$$

Proof. Performing steps analogous to those in the proof of Lemma 1 yields the above representations.

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